

THE E_6 STATE SUM INVARIANT OF LENS SPACES

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ABSTRACT. In this paper, we calculate the values of the E_6 state sum invariants for the lens spaces $L(p, q)$. In particular, we show that the values of the invariants are determined by $p \bmod 12$ and $q \bmod (p, 12)$. As a corollary, we show that the E_6 state sum is a homotopy invariant for the oriented lens spaces.

1. INTRODUCTION

In [5], Turaev and Viro constructed a state sum invariant of 3-manifolds based on their triangulations, by using the $6j$ -symbols of representations of the quantum group $U_q(\mathfrak{sl}_2)$. Further, Ocneanu [3] generalized the construction to the case of other types of $6j$ -symbols, say, the $6j$ -symbols of subfactors. *The E_6 state sum invariant* is the state sum invariant constructed from the $6j$ -symbols of the E_6 subfactor, which we denote by Z . Suzuki and Wakui [4] calculated the E_6 state sum invariant for some of the lens spaces, where they used the representation of the mapping class group of a torus $SL(2, \mathbb{Z})$.

In this paper, we calculate the E_6 state sum invariants for all of the lens spaces, as follows. For integers m, n , we denote by (m, n) the great common divisor of m and n . We put $\zeta = \exp(\pi\sqrt{-1}/12)$ and $[n] = (\zeta^n - \zeta^{-n})/(\zeta - \zeta^{-1})$ for an integer n , noting that

$$\begin{aligned} [12 - n] &= [n], & [n + 12] &= -[n], \\ [2] &= (1 + \sqrt{3})/\sqrt{2}, & [3] &= 1 + \sqrt{3}, & [4] &= (3 + \sqrt{3})/\sqrt{2}. \end{aligned}$$

Theorem 1.1. *For coprime integers p and q , the E_6 state sum invariant of the lens space $L(p, q)$ is given as*

$$(1.1) \quad Z(L(p, q)) = \begin{cases} |[p]| & \text{if } (p, 12) = 1, \\ [4][3]/[2] & \text{if } (p, 12) = 2, 6, \\ \zeta^{\pm 3}[4] & \text{if } (p, 12) = 3 \text{ and } q \equiv \pm 1 \pmod{3}, \\ 2\zeta^{\pm 2}[3] & \text{if } (p, 12) = 4 \text{ and } q \equiv \pm 1 \pmod{4}, \\ 2[4][3]/[2] & \text{if } 12|p \text{ and } q \equiv \pm 1 \pmod{12}, \\ 0 & \text{if } 12|p \text{ and } q \equiv \pm 5 \pmod{12}. \end{cases}$$

In particular, the value of $Z(L(p, q))$ is determined by $p \pmod{12}$ and $q \pmod{(p, 12)}$.

We note that we normalize the invariant so that $Z(S^3) = 1$. Thus, our Z is equal to wZ in [4], where we put $w = 2 + [3]^2 = 6 + 2\sqrt{3}$.

Corollary 1.2. *If there exists an orientation-preserving homotopy equivalence between the two lens spaces $L(p, q)$ and $L(p', q')$, then $Z(L(p, q)) = Z(L(p', q'))$.*

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2. THE CALCULATION OF THE E_6 STATE SUM INVARIANT

In this section, we briefly review the calculation of the E_6 state sum invariants for the lens spaces. Suzuki and Wakui [4] defined the representation $\rho : SL(2, \mathbb{Z}) \longrightarrow GL_{10}(\mathbb{C})$ by

$$\rho(S) = \frac{1}{w} \begin{pmatrix} 1 & [3] & 1 & [2]^2 & [3] & [3] & \frac{[4][3]}{[2]} & [3] & [3] & [2]^2 \\ [3] & \frac{[4][3]}{[2]}\sqrt{-1} & -[3] & -[3] & 0 & -\frac{[4][3]}{[2]}\sqrt{-1} & 0 & [3] & -[3] & [3] \\ 1 & -[3] & 1 & [2]^2 & -[3] & -[3] & -\frac{[4][3]}{[2]} & [3] & [3] & [2]^2 \\ [2]^2 & -[3] & [2]^2 & 1 & -[3] & -[3] & \frac{[4][3]}{[2]} & -[3] & -[3] & 1 \\ [3] & 0 & -[3] & -[3] & 0 & 0 & 0 & -2[3] & 2[3] & [3] \\ [3] & -\frac{[4][3]}{[2]}\sqrt{-1} & -[3] & -[3] & 0 & \frac{[4][3]}{[2]}\sqrt{-1} & 0 & [3] & -[3] & [3] \\ \frac{[4][3]}{[2]} & 0 & -\frac{[4][3]}{[2]} & \frac{[4][3]}{[2]} & 0 & 0 & 0 & 0 & 0 & -\frac{[4][3]}{[2]} \\ [3] & [3] & [3] & -[3] & -2[3] & [3] & 0 & [3] & [3] & -[3] \\ [3] & -[3] & [3] & -[3] & 2[3] & -[3] & 0 & [3] & [3] & -[3] \\ [2]^2 & [3] & [2]^2 & 1 & [3] & [3] & -\frac{[4][3]}{[2]} & -[3] & -[3] & 1 \end{pmatrix},$$

$$\rho(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\zeta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\zeta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta^8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta^{-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are generators of $SL(2, \mathbb{Z})$.

Let p, q be coprime integers. It is known, see [4, Lemma 4.2], that the E_6 state sum invariants of lens spaces are given as

$$(2.1) \quad Z(L(p, q)) = w^t \mathbf{e} \rho \left(\begin{pmatrix} -q & b \\ p & -a \end{pmatrix} \right) \mathbf{e},$$

where a, b are integers satisfying $aq - bp = 1$, and we put $\mathbf{e} = {}^t(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. We note that the right-hand side of the above formula does not depend on the choice of a and b .

3. PROOF OF THE THEOREM

In this section, we prove Theorem 1.1 and Corollary 1.2. In order to show Theorem 1.1, we show Proposition 3.3, which says that the values of $Z(L(p, q))$ have period 12 for p and q . In order to show Proposition 3.3, we show Lemmas 3.1 and 3.2, as follows.

Lemma 3.1. *Let p, q, p', q' be integers satisfying $(p, q) = 1$, $(p', q') = 1$ and $p \equiv p', q \equiv q' \pmod{12}$. Then, there exist integers a, b, a', b' such that*

$$aq - bp = 1, \quad a'q' - b'p' = 1 \quad \text{and} \quad a \equiv a', \quad b \equiv b' \pmod{12}.$$

Proof. We put integers a and b satisfying $aq - bp = 1$. Further, we put

$$\begin{pmatrix} a' & p' \\ b' & q' \end{pmatrix} = \begin{pmatrix} a & p \\ b & q \end{pmatrix} + 12 \begin{pmatrix} x & z \\ y & w \end{pmatrix},$$

noting that, by assumption, z and w are determined uniquely. It is sufficient to show that there exist integers x and y satisfying $a'q' - b'p' =$

1. The determinant of the right-hand side of the above formula is equal to

$$\begin{aligned}
 (3.1) \quad & (a + 12x)(q + 12w) - (p + 12z)(b + 12y) \\
 & = a(q + 12w) + 12xq' - (p + 12z)b - 12p'y \\
 & = 1 + 12(aw + xq' - zb - p'y).
 \end{aligned}$$

Since $(p', q') = 1$, there exists integers x and y satisfying

$$p'y - q'x = aw - zb.$$

Then, the last term of (3.1) is equal to 1. Thus, we have $a'q' - b'p' = 1$, as required. \square

We denote by I_n the n -by- n identity matrix. We put

$$\Gamma = \{P \in SL(2, \mathbb{Z}) \mid P \equiv I_2 \pmod{12}\}.$$

Lemma 3.2. $\Gamma \subset \ker \rho$, that is, $\rho(P) = I_{10}$ for any $P \in \Gamma$.

Proof. The GAP package Congruence [1] shows that Γ is a normal closure of the set of the following elements:

$$\begin{aligned}
 P_{1,\pm} &= \begin{pmatrix} 1 & \pm 12 \\ 0 & 1 \end{pmatrix}, & P_2 &= \begin{pmatrix} -143 & 12 \\ -12 & 1 \end{pmatrix}, & P_3 &= \begin{pmatrix} -155 & 84 \\ -24 & 13 \end{pmatrix}, \\
 P_4 &= \begin{pmatrix} -191 & 156 \\ -60 & 49 \end{pmatrix}, & P_5 &= \begin{pmatrix} -443 & 120 \\ -48 & 13 \end{pmatrix}, & P_6 &= \begin{pmatrix} -467 & 360 \\ -48 & 37 \end{pmatrix}, \\
 P_7 &= \begin{pmatrix} -299 & 108 \\ -36 & 13 \end{pmatrix}, & P_8 &= \begin{pmatrix} -311 & 216 \\ -36 & 25 \end{pmatrix}, & P_9 &= \begin{pmatrix} 937 & -396 \\ 168 & -71 \end{pmatrix}, \\
 P_{10} &= \begin{pmatrix} 157 & -36 \\ 48 & -11 \end{pmatrix}, & P_{11} &= \begin{pmatrix} 157 & -48 \\ 36 & -11 \end{pmatrix}, & P_{12} &= \begin{pmatrix} 205 & -84 \\ 144 & -59 \end{pmatrix}, \\
 P_{13} &= \begin{pmatrix} 157 & -72 \\ 24 & -11 \end{pmatrix}, & P_{14} &= \begin{pmatrix} 229 & -132 \\ 144 & -83 \end{pmatrix}, & P_{15} &= \begin{pmatrix} 169 & -108 \\ 36 & -23 \end{pmatrix}, \\
 P_{16} &= \begin{pmatrix} 181 & -132 \\ 48 & -35 \end{pmatrix}, & P_{17} &= \begin{pmatrix} 589 & -108 \\ 60 & -11 \end{pmatrix}, & P_{18} &= \begin{pmatrix} 649 & -384 \\ 120 & -71 \end{pmatrix}.
 \end{aligned}$$

Further, we can verify that these matrices are presented as the products of S and T , as follows.

$$\begin{aligned}
P_{1,\pm} &= T^{\pm 12}, & P_2 &= S^2 T^{12} S T^{12} S, \\
P_3 &= S^2 T^7 S T^2 S T^7 S T^2 S, & P_4 &= T^3 S T^{-5} S T^2 S T^{-4} S T S, \\
P_5 &= T^9 S T^{-4} S T^3 S T^4 S, & P_6 &= T^{10} S T^4 S T^3 S T^{-3} S T S, \\
P_7 &= T^8 S T^{-3} S T^4 S T^3 S, & P_8 &= T^9 S T^3 S T^4 S T^{-2} S T S, \\
P_9 &= T^5 S T^{-2} S T^{-4} S T^{-4} S T^{-3} S T^2 S, \\
P_{10} &= T^3 S T^{-4} S T^{-3} S T^4 S, & P_{11} &= T^4 S T^{-3} S T^{-4} S T^3 S, \\
P_{12} &= T S T^{-2} S T^3 S T^4 S T^{-2} S T^2 S, & P_{13} &= T^6 S T^{-2} S T^{-6} S T^2 S, \\
P_{14} &= T S T^{-2} S T^{-3} S T^4 S T^4 S T^2 S, & P_{15} &= S^2 T^4 S T^3 S T^{-3} S T^2 S T^2 S, \\
P_{16} &= T^4 S T^4 S T^{-3} S T^{-3} S T S, & P_{17} &= S^2 T^{10} S T^5 S T^{-2} S T^5 S, \\
P_{18} &= T^5 S T^{-2} S T^2 S T^{-4} S T^3 S T^2 S.
\end{aligned}$$

By using these formulae, we can verify that ρ takes to each of the matrices to I_{10} , completing the proof of the lemma. \square

Proposition 3.3. *Let p, q, p', q' be integers satisfying $(p, q) = 1$, $(p', q') = 1$ and $p \equiv p', q \equiv q' \pmod{12}$. Then, $Z(L(p, q)) = Z(L(p', q'))$.*

Proof. From Lemma 3.1, there exist matrices

$$A = \begin{pmatrix} -q & b \\ p & -a \end{pmatrix}, \quad A' = \begin{pmatrix} -q' & b' \\ p' & -a' \end{pmatrix} \in SL(2, \mathbb{Z})$$

such that $A \equiv A' \pmod{12}$. We put $P = I_2 + A^{-1}(A' - A)$. By definition, $P \in \Gamma$ and $A' = AP$. Thus, by (2.1) and Lemma 3.2, we have

$$Z(L(p', q')) = w^t \mathbf{e} \rho(A') \mathbf{e} = w^t \mathbf{e} \rho(A) \rho(P) \mathbf{e} = w^t \mathbf{e} \rho(A) \mathbf{e} = Z(L(p, q)),$$

completing the proof. \square

Proof of Theorem 1.1. By Proposition 3.3, the left-hand side of (1.1) has period 12 for p and q . On the other hand, the right-hand side of (1.1) also has period 12 for p and q . By [4, Appendix D], we can verify that (1.1) holds for coprime integers p and q with $1 \leq p \leq 12$, $q < p$. Therefore, (1.1) holds for any coprime integers p and q . \square

Proof of Corollary 1.2. It is known, see [2, 12.1], that there exists an orientation-preserving homotopy equivalence between the two lens spaces $L(p, q)$ and $L(p', q')$ if and only if $p = p'$ and $q \equiv n^2 q' \pmod{p}$ for some integer n .

When $(p, 12) \neq 3, 4, 12$, by Theorem 1.1 the value $Z(p, q)$ does not depend on q . Thus, $Z(L(p, q)) = Z(L(p', q'))$.

When $(p, 12) = k$ with $k = 3$ or 4 , we have $q \equiv n^2 q' \pmod{k}$ because $k|p$. Thus, $q \not\equiv -q' \pmod{k}$. Thus, by Theorem 1.1, $Z(L(p, q)) = Z(L(p', q'))$.

When $12|p$, we have $q \equiv n^2 q' \pmod{12}$. Thus, $q \not\equiv \pm 5q' \pmod{12}$. Thus, by Theorem 1.1, $Z(L(p, q)) = Z(L(p', q'))$. \square

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